All Square Roots of a König-Egerváry Graph

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June 16-19, 2015

Algorithmic Graph Theory on the Adriatic Coast 2015
Koper, Slovenia

Outline

1. Some definitions : independent sets, matchings

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2. König-Egerváry graphs

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6. Our findings ...

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7. Some open problems

Some definitions: independent sets

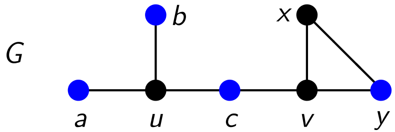
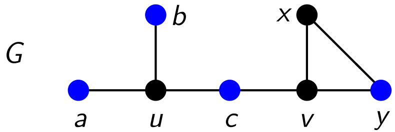


Figure: G has $\alpha(G) = |\{a, b, c, y\}| = 4$.

Definition

An independent or a stable set is a set of pairwise non-adjacent vertices.
The independence number or the stability number $\alpha(G)$ of G is the maximum cardinality of an independent set in G.

Some definitions: independent sets



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Example

$\{a\}, \{a, b\}, \{a, b, x\}, \{a, b, c, y\}$ are independent sets of G.

$\{a, b, c, x\}, \{a, b, c, y\}$ are maximum independent sets, hence $\alpha(G) = 4$.

Some definitions: matchings and matching number

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A matching in G is a set of non-incident edges.

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Example

$\{a_1 a_2\}$ is a maximum matching in K_3, hence $\mu(K_3) = 1$
$\{v_1 v_2, v_3 v_4\}$ is maximum matching in C_5, hence $\mu(C_5) = 2$
$\{t_1 t_2, t_3 t_4, t_5 t_5\}$ is maximum matching in G, hence $\mu(G) = 3$

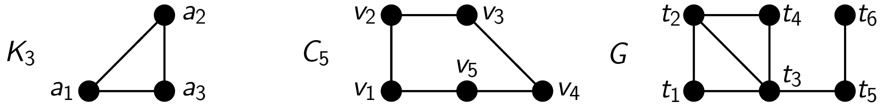

Figure: Only G has perfect matchings; e.g., $M = \{t_1 t_3, t_2 t_4, t_5 t_6\}$.

Some definitions: König-Egerváry graphs

Remark

$\lfloor |V|/2 \rfloor + 1 \leq \alpha(G) + \mu(G) \leq |V|$ *hold for every graph* $G = (V, E)$.

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Definition (R. W. Deming (1979), F. Sterboul (1979))

$G = (V, E)$ *is a König–Egerváry graph if* $\alpha(G) + \mu(G) = |V|$.

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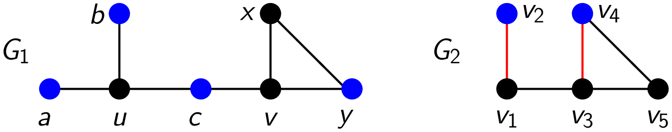

Figure: G_1 is a König–Egerváry graph, since $\alpha(G_1) + \mu(G_1) = 7 = |V(G_1)|$, while G_2 is **not** a König–Egerváry graph, as $\alpha(G_2) + \mu(G_2) = 4 < 5 = |V(G_2)|$.

Theorem (D. König (1931), E. Egerváry (1931))

Each bipartite graph $G = (V, E)$ *satisfies* $\alpha(G) + \mu(G) = |V|$.

A characterization for König-Egerváry graphs

Notation

If $A \cap B = \emptyset$ in $G = (V, E)$, then $(A, B) = \{ab \in E : a \in A, b \in B\}$.
If $S \in Ind(G)$ and $H = G - S$, we write $G = S * H$.

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Theorem (Levit and Mandrescu, Discrete Math. 2003)

For a graph $G = (V, E)$, the following properties are equivalent:

(i) G is a König-Egerváry graph;

(ii) $G = S * H$, where $S \in \Omega(G)$ and $|S| \geq \mu(G) = |V - S|$;

(iii) $G = S * H$, where S is an independent set with $|S| \geq |V - S|$ and $(S, V - S)$ contains a matching of size $|V - S|$.

$S = \{x, y\} \in \Omega(G)$

$\mu(G) = 2 < |V - S|$

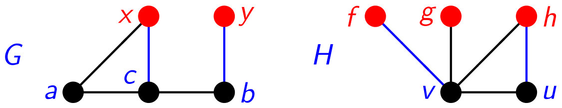

Figure: By above theorem, part (ii), only H is a König-Egerváry graph.

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$S = \{x, y\}$

$M = \{ac, yb\}$

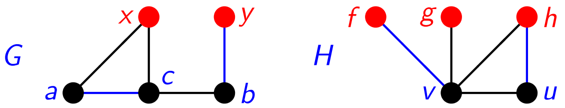
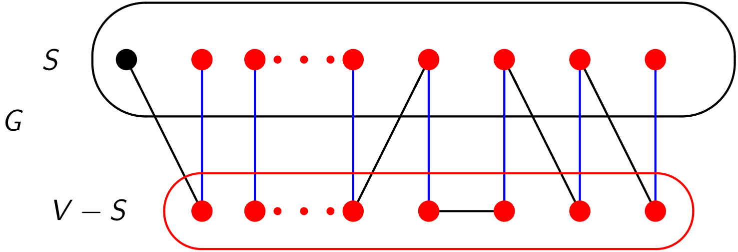


Figure: $M \not\subseteq (S, V(G) - S)$, hence G is not a König-Egerváry graph. H is a König-Egerváry graph.



- A König-Egerváry graph $G = S * H$

Corona of graphs

Definition

The **corona** of the graphs X and $\{H_i : 1 \leq i \leq n\}$ is the graph $G = X \circ \{H_1, H_2, ..., H_n\}$ obtained by joining each $v_i \in V(X)$ to all the vertices of H_i, where $V(X) = \{v_i : 1 \leq i \leq n\}$.
If every $H_i = H$, we write $G_1 = X \circ H$.

- $G = H \circ K_1$ is a **König-Egerváry graph** with a **perfect matching**.

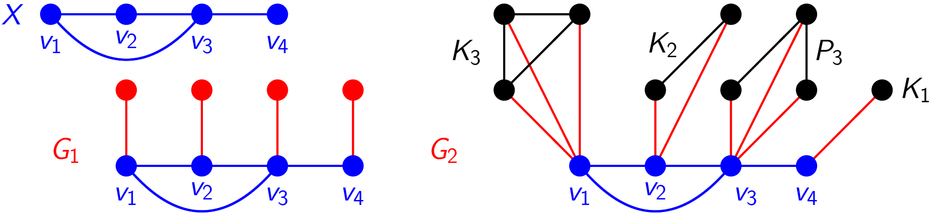


Figure: The graphs $G_1 = X \circ K_1$ and $G_2 = X \circ \{K_3, K_2, P_3, K_1\}$.

Square of a graph

Definition

The square of the graph $G = (V, E)$ is the graph $G^2 = (V, U)$, where $xy \in U$ if and only if $x \neq y$ and $dist_G(x, y) \leq 2$.

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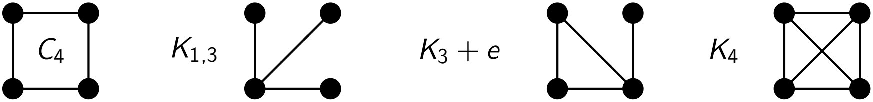

Figure: Non-isomorphic graphs having the same square.

Example

$$C_4^2 = K_{1,3}^2 = (K_3 + e)^2 = K_4^2 = K_4.$$

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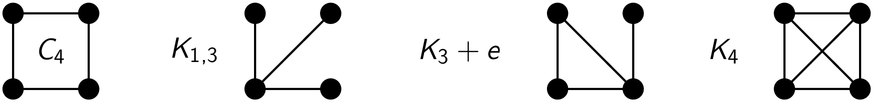


Figure: Non-isomorphic graphs having the same square.

Example

$$C_4^2 = K_{1,3}^2 = (K_3 + e)^2 = K_4^2 = K_4.$$

Remark

(i) There is no G such that $G^2 = C_4$.
(ii) If one of the n vertices of G has $n - 1$ neighbors, then $G^2 = K_n$.

Square root of a graph

Definition

If there is some graph H such that $H^2 = G$,

then H is called a **square root** of G, i.e., $H \in \sqrt{G}$.

- A graph may have **more** than one square root.

Example

Every H of order n that has a vertex of degree $n-1$ is a root of K_n.

- There are graphs having **no** square root.

Example

P_3 has **no** square root, i.e., the equation $H^2 = P_3$ has **no** solution.

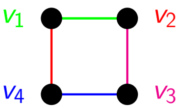
Theorem (A. Mukhopadhyay, J. Combin. Th., 1967)

A connected graph G on n vertices $v_1, v_2, \ldots, v_n$, has a square root **if and only if** *there exists an* **edge clique cover** $Q_1, \ldots, Q_n$ *of G such that, for all $i, j \in \{1, \ldots, n\}$, the following hold:*

(i) *Q_i contains v_i, for all $i \in \{1, \ldots, n\}$; and*

(ii) *for all $i, j \in \{1, \ldots, n\}$, Q_i contains v_j* **iff** *Q_j contains v_i.*

C_4



$Q_1 = \{v_1, v_2\}$ $\quad$ $Q_2 = \{v_2, v_3\}$

$Q_3 = \{v_3, v_4\}$ $\quad$ $Q_4 = \{v_4, v_1\}$

$\{Q_1, Q_2, Q_3, Q_4\}$ is the only edge clique cover of C_4

Figure: C_4 has no square root: $v_3 \in Q_2$, while $v_2 \notin Q_3$.

Some old results

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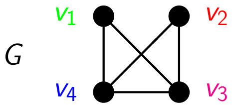
Example

The edge clique cover Q_1, Q_2, Q_3, Q_4 satisfies **(i)** and **(ii)**.



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P_4 is a square root of G

Some algorithmic results

Theorem (D.J. Ross and F. Harary, Bell System Tech. J., 1960)

Tree roots *of a graph, when they exist, are unique up to isomorphism.*

Theorem (Y. L. Lin, S. Skiena, LNCS 557, 1991)

There is an $O(m)$ time algorithm for finding the square roots of a planar graph.

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The tree square root of a graph can be found in $O(m)$ time, where m denotes the number of edges of the given tree square root.

Theorem (Y. L. Lin, S. Skiena, SIAM J. of Discrete Math, 1995)

There is a linear time algorithm to recognize squares G^2 of graphs, where G is a tree.

Problem (SqR)

A Square Root of a Graph
Instance: *A graph G.*
Question: *Does there exist a graph H such that $H^2 = G$?*

Theorem (R. Motwani, M. Sudan, Discrete Applied Math, 1994)

Problem **SqR** *is* **NP***-complete.*

Theorem (L.C. Lau, D.G. Corneil, SIAM J. Discrete Math, 2004)

The Problem **SqR** *remains* **NP***-complete for chordal graphs.*

Theorem (Martin Milanič, Oliver Schaudt, Discrete Applied Math, 2013)

The Problem **SqR** *is polynomial for trivially perfect and threshold graphs.*

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The Problem **SqR** *remains* **NP**-*complete for chordal graphs.*

- A **chordal** graph is one whose cycles on $q \geq 4$ vertices have a chord.

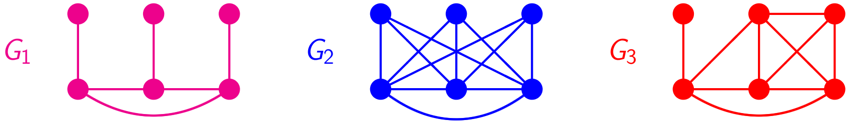

Figure: Chordal graphs: only G_2 has square roots, namely $G_1 \in \sqrt{G_2}$.

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Instance: *A graph G.*
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Theorem (Martin Milanič, Oliver Schaudt, Discrete Applied Math, 2013)

The Problem **SqR** *is polynomial for trivially perfect graphs.*

- G is a **trivially perfect graph** if each of its induced subgraphs H has $\alpha(H)$ maximal cliques (M. C. Golumbic, Discrete Math. 1978).
- They are exactly the (P_4 and C_4)-free graphs (Golumbic, DM 1978).

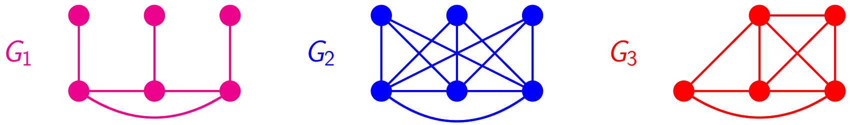

Figure: Only G_2, G_3 are trivially perfect, and $G_1 \in \sqrt{G_2}$

Threshold graphs

Definition

A graph $G = (V, E)$ is called threshold (V. Chvatal and P. L. Hammer, 1977) if it can be obtained from K_1 by iterating, in any order, the operations of adding a new vertex which is connected to

no other vertex (i.e., an isolated vertex) or

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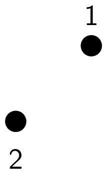

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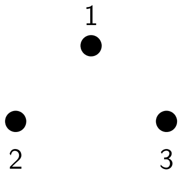

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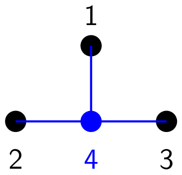

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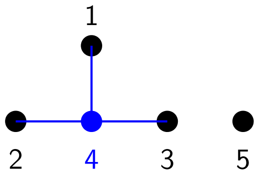

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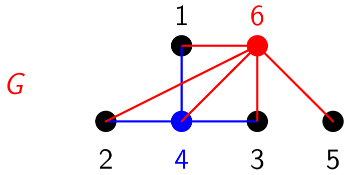


Figure: G is a threshold graph : 4 and 6 are dominating vertices.

Problem (SqR)

A Square Root of a Graph

Instance: *A graph G.*

Question: *Does there exist a graph H such that $H^2 = G$?*

Theorem (Martin Milanič, Oliver Schaudt, Discrete Applied Math, 2013)

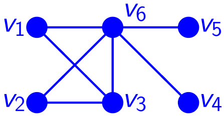
The Problem **SqR** *is polynomial for threshold graphs.*

- Threshold graphs are exactly the (P_4 and C_4 and $2K_2$)-free graphs (V. Chvatal, P. L. Hammer, 1977).

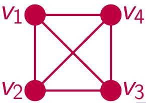
Examples

G_1 and G_2 are threshold graphs, but only G_2 has square roots.

G_1



G_2



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3. How to compute all **square roots** of a **König-Egerváry** graph?

Example

- The graph G_2 has G_1 as a square root, i.e., $G_1 \in \sqrt{G_2}$.
- G_3 has no square roots, because it has a leaf.

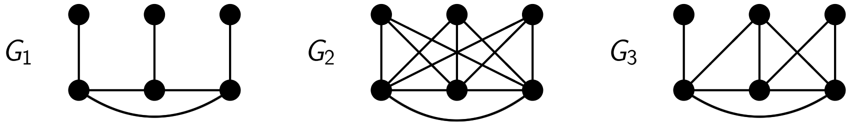

Figure: König-Egerváry graphs.

Theorem

If a connected König-Egerváry graph G of order ≥ 3 has a square root, then G has **perfect matchings** *and a* **unique** *maximum independent set.*

Example

The graph G_2 has G_1 as a square root.

G_3 has no square roots, because it has a leaf.

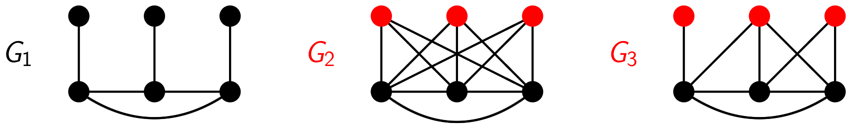

Figure: König-Egerváry graphs.

- The converse of theorem above is **not** necessarily true; e.g., G_3.

Squares, roots and König-Egerváry graphs

Theorem (L & M, Graphs and Combinatorics, 2013)

For a graph H of order $n \geq 2$, the following are equivalent:

(i) *H^2 is a König-Egerváry graph;*

(ii) *H has a perfect matching consisting of pendant edges.*

Corollary

Each square root of a König-Egerváry graph G, if any, is of the form $H_0 \circ K_1$ for some graph H_0.

Figure: König-Egerváry graphs: $H = H_0 \circ K_1$ and H^2.

- There are König-Egerváry graphs, whose squares are not König-Egerváry graphs. E.g., every C_{2n}.
- There are non-König-Egerváry graphs, whose squares are not König-Egerváry graphs. E.g., every C_{2n+1}.

Theorem (L & M, Graphs and Combinatorics, 2013)

For a graph H of order $n \geq 2$, the following are equivalent:

(i) H^2 *is a König-Egerváry graph;*

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Each square root of a König-Egerváry graph G, if any, is of the form $H_0 \circ K_1$ for some graph H_0.

Simplicial graphs

Definition (G. H. Cheston, E. O. Hare, S. T. Hedetniemi and R. C. Laskar, Congressus Numer 67, 1988)

A vertex v is simplicial in G if $N_G(x)$ is a **clique**. A **simplex** is a clique containing at least one simplicial vertex. G is a **simplicial graph** if each of its vertices is either simplicial or adjacent to a simplicial vertex.

Theorem (Cheston et al., Congressus Numer 67, 1988)

If G is a simplicial graph and $Q_1, \ldots, Q_q$ are the simplices of G, then $V(G) = \cup\{V(Q_i) : 1 \leq i \leq q\}$ and $q = \alpha(G)$.

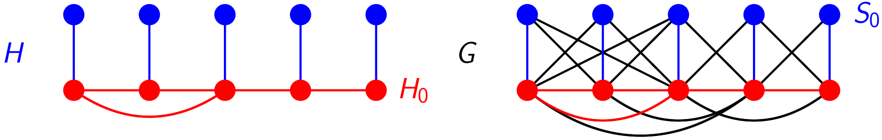


Figure: König-Egerváry graphs: $H = H_0 \circ K_1$ and $G = H^2$.

Square root of a König-Egerváry graph

- A vertex v is simplicial in G if its neighborhood $N_G(v)$ is a **clique**.
- G is **simplicial** if each of its vertices is either simplicial or adjacent to a simplicial vertex.

Theorem

If a König-Egerváry graph G, of order $n \geq 3$, has a square root, then every vertex of its unique maximum independent set, say S_0, is **simplicial**. *Moreover, $\{N_G(x) : x \in S_0\}$ is an* **edge clique cover** *of $G[V(G) - S_0]$.*

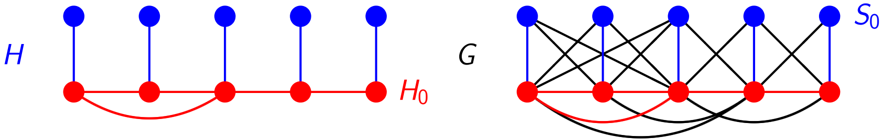


Figure: König-Egerváry graphs: $H = H_0 \circ K_1$ and $G = H^2$.

Square roots of a König-Egerváry graph

Corollary

Each square root of a König-Egerváry graph G, if any, is of the form $H_0 \circ K_1$ for some graph H_0.

Theorem

If a König-Egerváry graph G, of order $n \geq 3$, has a square root, then every vertex of its unique maximum independent set, say S_0, is **simplicial**. *Moreover, $\{N_G(x) : x \in S_0\}$ is an* **edge clique cover** *of $G[V(G) - S_0]$.*

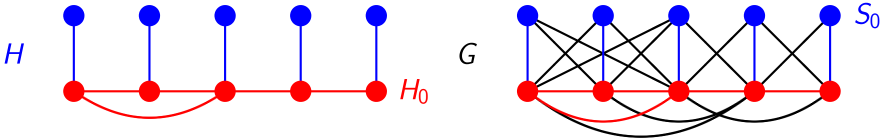


Figure: König-Egerváry graphs: $H = H_0 \circ K_1$ and $G = H^2$.

Problem (AllSqR)

All Square Roots of a König-Egerváry Graph

Instance: *A connected König-Egerváry graph* G.

Output: *All graphs* H *such that* $H^2 = G$.

Theorem

Problem **AllSqR** *is solvable in*

$$O\left(\left(|E| \cdot |V| + |V|^2\right) + ((\Delta(G) + M(n)) \cdot |V| \cdot per(G))\right)$$

time, where $per(G)$ *is the number of perfect matchings of* $G = (V, E)$, *and* $M(n)$ *is the time complexity of a matrix multiplication for two* $n \cdot n$ *matrices.*

Core of a graph

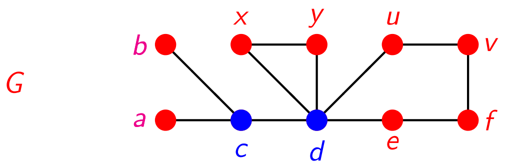
Definition (Levit and Mandrescu, Discrete Applied Math, 2002)

$\mathrm{core}(G)$ is the intersection of all maximum independent sets of G.

- The problem of whether $\mathrm{core}(G) \neq \emptyset$ is **NP**-complete (**Endre Boros**, M. C. Golumbic, V. E. Levit, Discrete Applied Math, 2002).

Fact

G has a **unique** *maximum independent set* **if and only if** $\mathrm{core}(G)$ *is a* **maximum** *independent set.*



Checking whether a K-E graph may have a square root

- Testing whether a graph has a **unique** maximum independent set is **NP**-hard (A. Pelc, IEEE Transactions on computers, 1991).
- We need to check whether a König-Egerváry graph with **perfect matchings** has a **unique** maximum indep set, and if positive, to find it.

Lemma

Let $G = (V, E)$ be a König-Egerváry graph having a **perfect matching**, *and $v \in V$. Then the following assertions are true:*

(i) $v \in \mathrm{core}(G)$ **iff** $G - v$ *is* **not** *a König-Egerváry graph;*

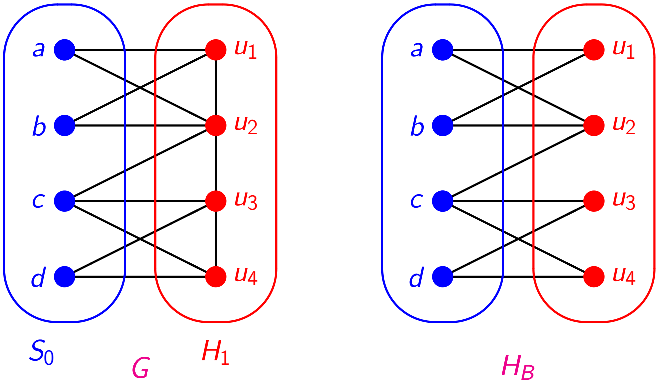
(ii) one can find $\mathrm{core}(G)$ *in* $O(|V| \cdot |E| + |V|^2)$ *time;*

(iii) one can check whether G has a **unique** *maximum independent set (namely* $\mathrm{core}(G)$*), and find it, in* $O(|V| \cdot |E| + |V|^2)$ *time.*

A sketch of an algorithm generating all square roots of a K-E graph

- There is a poly time algorithm finding a maximum matching M in G that needs $O(|E| \cdot \sqrt{|V|})$ time (V. V. Vazirani, Combinatorica 1994).
- If $2|M| \neq |V|$, i.e., M is **not** perfect, then G has **no** square root.
- Assume that M is a perfect matching. Hence $\alpha(G) = \mu(G) = |M|$.
- Since G is a König-Egerváry graph with a perfect matching, one can find $S_0 = \mathrm{core}(G)$ in time $O(|V| \cdot |E| + |V|^2)$.
- If $\alpha(G) \neq |S_0|$, then G has no square root, since it has more than one maximum independent set.

- Otherwise, we infer that $\Omega(G) = \{S_0\}$ and $G = S_0 * H_1$.



- One can run an algorithm generating all perfect matchings in the bipartite graph $H_B = (S_0, V(H_1), E - E(H_1))$ with the time complexity $O\left(\sqrt{|V|} \cdot |E(H_B)| + per(H_B) \cdot \log |V|\right)$ (T. Uno, LNCS **2223**, 2001).

- In other words, every solution of the equation $G = H^2$ is based on a choice of a perfect matching of the bipartite graph H_B.
- Let $M_0 = \{x_i y_i : 1 \leq i \leq |V|/2\}$ be such a perfect matching of H_B, where $S_0 = \{x_i : 1 \leq i \leq |V|/2\}$.

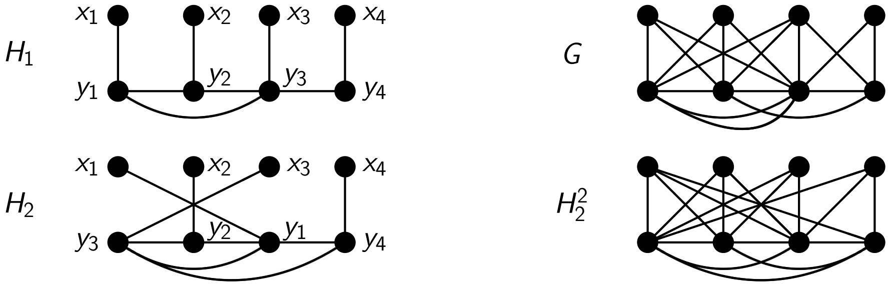


Figure: H_1, H_2 are candidates for the equation $H^2 = G$, corresponding to different perfect matchings of H_B, but only $H_1^2 = G$.

To define the edge set of the graph H as a function of the perfect matching M_0, we proceed as follows:

- keep M_0 be such as a part of $E(H)$;
- check that for every $x_k z \in E(G) - \{x_k y_k\}, 1 \leq k \leq |V|/2$, there exists the edge $y_k z \in E(G)$, otherwise M_0 may not generate a square root of G;
- build the graph H_0 as follows:

$$\begin{aligned} V(H_0) &= V - S_0, \\ E(H_0) &= \{y_k z : x_k z \in E(G) - \{x_k y_k\}, 1 \leq k \leq |V|/2\}; \end{aligned}$$

- if $(V, E(H_0) \cup M_0)^2 = G$, then the graph $(V, E(H_0) \cup M_0)$ is a square root of G, otherwise M_0 does not generate a square root.

- Since S_0 is the unique maximum independent set of a König-Egerváry graph G, and, on the other hand, by a Theorem characterizing König-Egerváry graphs, every matching of G is contained in $(S_0, V - S_0)$, one may conclude that the graphs $G = S_0 * H_1$ and $H_B = (S_0, V(H_1), E - E(H_1))$ have the same perfect matchings.
- In summary, testing all the perfect matchings of the bipartite graph H_B one can generate $\sqrt{G}$ with

$$O\left(\left(|E| \cdot |V| + |V|^2\right) + ((\Delta(G) + M(n)) \cdot |V| \cdot per(G))\right)$$

time complexity.

Symmetric bipartite graphs

Definition (N. Kakimura, 2008)

A bipartite graph $G = (A, B, E)$ with $|A| = |B|$ is said to be **symmetric** if $a_j b_i \in E$ holds for every $a_i b_j \in E$.

Example

G_1 is bipartite and symmetric, while

G_2 is bipartite, but not symmetric.

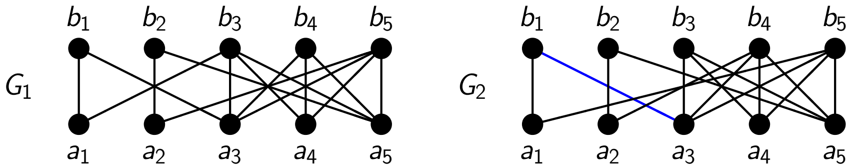


Figure: Bipartite graphs on the same vertices.

Definition

Let $F = (A, B, E)$ be a bipartite graph, such that $A = \{a_j : 1 \leq j \leq p\}$ and $B = \{b_k : 1 \leq k \leq q\}$. The **adjacency matrix** of F is $Adj(F) = (x_{jk})_{p \times q}$, where $x_{jk} = 1$ if $a_j b_k \in E$, and $x_{jk} = 0$, otherwise.

Example

$$Adj(F) = \begin{array}{c} \left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \right\} \end{array} \begin{pmatrix} \mathbf{1} & 1 & 0 & 0 & 0 \\ 1 & 1 & \mathbf{1} & 0 & 0 \\ 0 & 1 & 1 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 1 & 1 & 1 \end{pmatrix}$$ and M is a maximum matching.

F

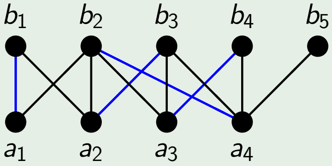


Figure: "Blue matching" is a **maximum matching**.

Definition

Let M be a perfect matching of $F = (A, B, E)$. The **permutation matrix** P_M determined by M is: $P_M(i,j) = 1$ **if and only if** $a_j b_i \in M$.

Example

$$Adj(F) = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \\ \end{array} \begin{pmatrix} \mathbf{1} & 1 & 0 & 0 \\ 1 & 1 & \mathbf{1} & 0 \\ 0 & 1 & 1 & \mathbf{1} \\ 0 & \mathbf{1} & 1 & 1 \end{pmatrix} \Longrightarrow P_M = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \end{pmatrix}$$

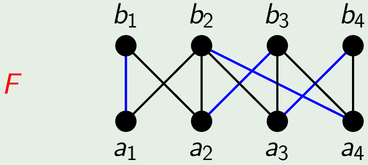


Figure: $M = \{a_1 b_2, a_2 b_3, a_3 b_4, a_4 b_2\}$ is a **perfect matching**.

Definition

Let M be a perfect matching of $F = (A, B, E)$.
The **permutation matrix** P_M determined by M is:

$$P_M(i, j) = 1 \textbf{ if and only if } a_j b_i \in M.$$

Example

$$Adj(F) = \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{pmatrix} \mathbf{1} & 1 & 0 & 0 \\ 1 & 1 & \mathbf{1} & 0 \\ 0 & 1 & 1 & \mathbf{1} \\ 0 & \mathbf{1} & 1 & 1 \end{pmatrix} \Longrightarrow P_M = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \end{pmatrix}$$

F

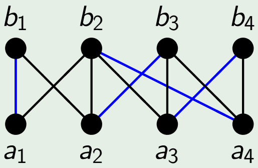


Figure: $M = \{a_1 b_2, a_2 b_3, a_3 b_4, a_4 b_2\}$ is a **perfect matching**.

Definition

Let M is a perfect matching of $F = (A, B, E)$. The corresponding **adjacency matrix** of F with **respect to** M is $Adj(F, M) = Adj(F) * P_M$.

Example

$$Adj(F, M) = \begin{pmatrix} \mathbf{1} & 1 & 0 & 0 \\ 1 & 1 & \mathbf{1} & 0 \\ 0 & 1 & 1 & \mathbf{1} \\ 0 & \mathbf{1} & 1 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 1 \\ 1 & \mathbf{1} & 0 & 1 \\ 0 & 1 & \mathbf{1} & 1 \\ 0 & 1 & 1 & \mathbf{1} \end{pmatrix}$$

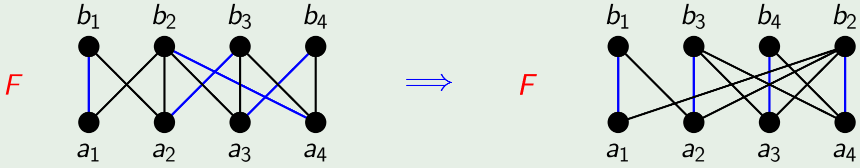


Figure: "Blue matching" is a **perfect matching**.

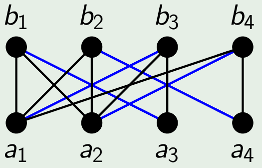
Definition

A perfect matching M of $F = (A, B, E)$ is **symmetric** if $Adj(F, M) = Adj(F) * P_M$ *is* **symmetric**, i.e., $M = \left\{ a_i b_{\tau(i)} : 1 \leq i \leq |A| \right\}$ is **symmetric** if $a_{\tau^{-1}(j)} b_{\tau(i)} \in E$ holds for every $a_i b_j \in E$.

Example

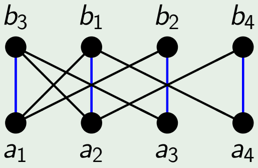
$$Adj(F, M) = \begin{pmatrix} 1 & 0 & \mathbf{1} & 1 \\ 0 & 1 & 1 & \mathbf{1} \\ \mathbf{1} & 0 & 1 & 0 \\ 0 & \mathbf{1} & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 1 & 1 & 0 \\ 1 & \mathbf{1} & 0 & 1 \\ 1 & 0 & \mathbf{1} & 0 \\ 0 & 1 & 0 & \mathbf{1} \end{pmatrix}$$

F



$\Longrightarrow$

F



Definition

Let $F = (A, B, E)$ be a bipartite graph, such that $A = \{a_j : 1 \leq j \leq p\}$ and $B = \{b_k : 1 \leq k \leq q\}$. The adjacency matrix of F is $Adj(F) = (x_{jk})_{p \times q}$, where $x_{jk} = 1$ if $a_j b_k \in E$ and $x_{jk} = 0$, otherwise.

Definition

Let $F = (A, B, E)$ be a bipartite graph, and M be a perfect matching of F. The corresponding adjacency matrix of F with respect to M is $Adj(F, M) = Adj(F) * P_M$.

Clearly, if $F = (A, B, E)$ has a perfect matching M,then $Adj(F, M)$ has $x_{kk} = 1$, for all $k \in \{1, 2, ..., |A|\}$.

Definition

Let $F = (A, B, E)$ be a bipartite graph. A perfect matching M is *symmetric* if $Adj(F, M)$ *is symmetric.* In other words a perfect matching $M = \left\{a_i b_{\tau(i)} : 1 \leq i \leq |A|\right\}$ is *symmetric* if $a_{\tau^{-1}(j)} b_{\tau(i)} \in E$ holds for every $a_i b_j \in E$.

Definition

A perfect matching $M = \left\{ a_i b_{\tau(i)} : 1 \leq i \leq |A| \right\}$ in $F = (A, B, E)$ is *symmetric* if $a_{\tau^{-1}(j)} b_{\tau(i)} \in E$ holds for every $a_i b_j \in E$.

- A bipartite graph may have both symmetric and non-symmetric perfect matchings.

Example

$M_1 = \{a_i b_i : 1 \leq i \leq 5\}$ and $M_2 = \{a_1 b_1, a_2 b_2, a_3 b_4, a_4 b_5, a_5 b_3\}$ are perfect matchings, but only M_1 is **symmetric**.

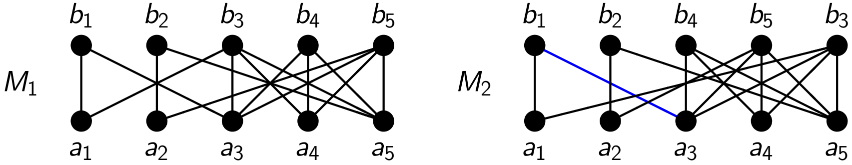


Figure: Both M_1 and M_2 are perfect matchings of the same bipartite graph.

Definition

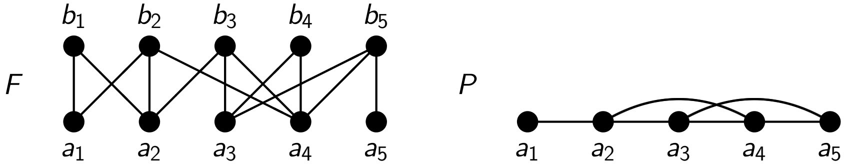
The projection of $F = (A, B, E)$ on A with respect to a perfect matching $M = \{a_i b_i : 1 \leq i \leq |A|\}$ is a graph $P = P(F, M, A)$ defined as follows:

$V(P) = A$ and $E(P) = \{a_i a_j : a_i b_j \in E \text{ or } a_j b_i \in E\}$.

Example

The projection $P = P(F, M, A)$ of $F = (A, B, E)$ on A

with respect to the perfect matching $M = \{a_i b_i : 1 \leq i \leq 5\}$.



Theorem (A. Mukhopadhyay, J. Combin. Th., 1967)

A connected graph G on n vertices $v_1, v_2, \ldots, v_n$, has a square root **if and only if** *there exists an* **edge clique cover** $Q_1, \ldots, Q_n$ *of G such that, for all $i, j \in \{1, \ldots, n\}$, the following hold:*

(i) *Q_i contains v_i, for all $i \in \{1, \ldots, n\}$; and*

(ii) *for all $i, j \in \{1, \ldots, n\}$, Q_i contains v_j* **iff** *Q_j contains v_i.*

I.e., the fact that G has a square root means that a natural matching $\{v_i Q_i : 1 \le i \le n\}$ in the **vertex-clique bipartite graph** is symmetric.

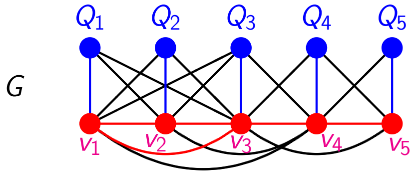


Figure: Vertex-clique bipartite graph.

Square roots of a König-Egerváry graph

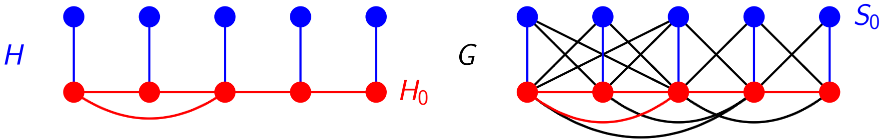
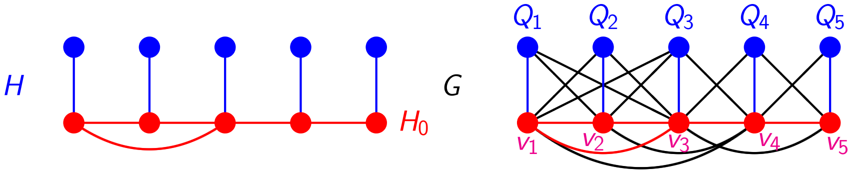


Figure: König-Egerváry graphs: $H = H_0 \circ K_1$ and $G = H^2$.

$Q_1 = \{v_1, v_2, v_3\}$, $Q_2 = \{v_1, v_2, v_3\}$,
$Q_3 = \{v_1, v_2, v_3, v_4\}$, $Q_4 = \{v_3, v_4, v_5\}$, $Q_5 = \{v_4, v_5\}$



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Theorem

If a König-Egerváry graph G, of order $n \geq 3$, has a square root, then every vertex of its unique maximum independent set, say S_0, is **simplicial**. *Moreover, $\{N_G(x) : x \in S_0\}$ is an* **edge clique cover** *of $G[V(G) - S_0]$.*

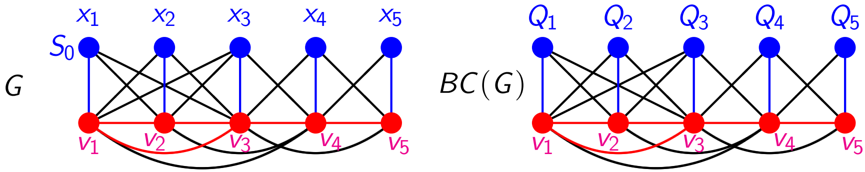

Figure: A König-Egerváry graph G and its vertex-clique bipartite graph $BC(G)$.

Definition (Double Covering)

Let $G = (V, E)$, $V = \{v_1, v_2, ..., v_n\}$, and $\hat{V} = \{\hat{v}_1, \hat{v}_2, ..., \hat{v}_n\}$. The **double covering** of G is the bipartite graph $B(G)$ with the bipartition $\{V, \hat{V}\}$ and edges $v_i\hat{v}_j$ and $v_j\hat{v}_i$ for every edge $v_iv_j \in E$.

Theorem (R. A. Brualdi, F. Harary, Z. Miller, J. Graph Theory, 1980)

$B(G)$ *is connected* **iff** G *is connected and non-bipartite.*

Theorem (Dragan Marusic, R. Scapellato, N. Zagagha Salvi)

Let A be a g-matrix (a square symmetric $(0, 1)$ matrix with the 0 (zero) principal diagonal) of order n , and R be a permutation matrix representing an n-cycle. Then $A * R$ is a g-matrix if and only if $A = 0$.

Theorem

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Proof.

A Latin Square Sketch of the Proof:

$$\begin{matrix} 1 & * & * & * \\ * & 1 & * & * \\ * & * & 1 & * \\ * & * & * & 1 \end{matrix} \Longrightarrow \begin{matrix} 1 & * & * & 2 \\ 2 & 1 & * & * \\ * & 2 & 1 & * \\ * & * & 2 & 1 \end{matrix} \Longrightarrow \begin{matrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{matrix}$$

$\square$

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$$\begin{array}{cccc} 1 & * & * & * \\ * & 1 & * & * \\ * & * & 1 & * \\ * & * & * & 1 \end{array} \Longrightarrow \begin{array}{cccc} 1 & * & * & 2 \\ 2 & 1 & * & * \\ * & 2 & 1 & * \\ * & * & 2 & 1 \end{array} \Longrightarrow \begin{array}{cccc} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{array}$$

□

- $$\begin{array}{ccccc} & y_1 & y_2 & y_3 & y_4 \\ x_1 & \mathbf{1} & 1 & 0 & 1 \\ x_2 & 1 & \mathbf{1} & 1 & 0 \\ x_3 & 0 & 1 & 1 & \mathbf{1} \\ x_4 & 1 & 0 & \mathbf{1} & 1 \end{array} \qquad \begin{array}{ccccc} & y_1 & y_2 & y_4 & y_3 \\ x_1 & \mathbf{1} & 1 & 1 & 0 \\ x_2 & 1 & \mathbf{1} & 0 & \mathbf{1} \\ x_3 & 0 & 1 & \mathbf{1} & 1 \\ x_4 & 1 & \mathbf{0} & 1 & \mathbf{1} \end{array}$$

Theorem

If $F = (A, B, E)$ has symmetric perfect matchings and twins, then it has at least two symmetric perfect matchings.

Proof.

Let $M = \{a_i b_i : 1 \leq i \leq q\}$ be a symmetric perfect matching of F, and b_j, b_k be twins.

Then the columns of the matrix $Adj(F, M)$, corresponding to b_j and b_k, are identical.

Thus interchanging these two columns leaves the matrix symmetric.

Hence the principal diagonal of the new matrix defines another perfect matching, that is symmetric, as well. □

Remark

If $F = (A, B, E)$ has no twins, then it may have more than one symmetric perfect matching.

Example

$G = (A, B, E)$ has no twins, while the perfect matchings

$M_1 = \{a_i b_i : 1 \leq i \leq 4\}$,

$M_2 = \{a_1 b_2, a_2 b_1, a_3 b_4, a_4 b_3\}$

$M_3 = \{a_1 b_3, a_2 b_4, a_3 b_1, a_4 b_2\}$ are symmetric.

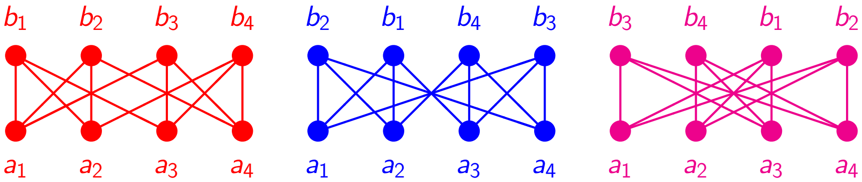

Figure: A bipartite graph $G = (A, B, E)$ and three of its perfect matchings.

- $Perm \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} = 9$

- $$Perm\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} = 9$$

- $$\begin{array}{ccccc} \mathbf{1} & y_1 & y_2 & y_3 & y_4 \\ x_1 & \mathbf{1} & 1 & 0 & 1 \\ x_2 & 1 & \mathbf{1} & 1 & 0 \\ x_3 & 0 & 1 & \mathbf{1} & 1 \\ x_4 & 1 & 0 & 1 & \mathbf{1} \end{array}$$

- $$Perm\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} = 9$$

- $$\begin{array}{ccccc} \mathbf{1} & y_1 & y_2 & y_3 & y_4 \\ x_1 & \mathbf{1} & 1 & 0 & 1 \\ x_2 & 1 & \mathbf{1} & 1 & 0 \\ x_3 & 0 & 1 & \mathbf{1} & 1 \\ x_4 & 1 & 0 & 1 & \mathbf{1} \end{array}$$

- $$\begin{array}{ccccc} 2 & y_1 & y_2 & y_3 & y_4 \\ x_1 & \mathbf{1} & 1 & 0 & 1 \\ x_2 & 1 & \mathbf{1} & 1 & 0 \\ x_3 & 0 & 1 & 1 & \mathbf{1} \\ x_4 & 1 & 0 & \mathbf{1} & 1 \end{array} \quad \begin{array}{ccccc} 2 & y_1 & y_2 & y_4 & y_3 \\ x_1 & \mathbf{1} & 1 & 1 & 0 \\ x_2 & 1 & \mathbf{1} & 0 & \mathbf{1} \\ x_3 & 0 & 1 & \mathbf{1} & 1 \\ x_4 & 1 & \mathbf{0} & 1 & \mathbf{1} \end{array}$$

-

4	y_1	y_2	y_3	y_4	4	y_2	y_3	y_4	y_1
x_1	1	**1**	0	1	x_1	**1**	**0**	1	0
x_2	1	1	**1**	0	x_2	**1**	**1**	0	1
x_3	0	1	1	**1**	x_3	1	1	**1**	1
x_4	**1**	0	1	1	x_4	0	1	1	**1**

4	y_1	y_2	y_3	y_4	4	y_2	y_3	y_4	y_1
x_1	1	**1**	0	1	x_1	**1**	**0**	1	0
x_2	1	1	**1**	0	x_2	**1**	**1**	0	1
x_3	0	1	1	**1**	x_3	1	1	**1**	1
x_4	**1**	0	1	1	x_4	0	1	1	**1**

5	y_1	y_2	y_3	y_4	5	y_2	y_1	y_3	y_4
x_1	1	**1**	0	1	x_1	**1**	1	0	**1**
x_2	**1**	1	1	0	x_2	1	**1**	1	0
x_3	0	1	**1**	1	x_3	1	0	**1**	1
x_4	1	0	1	**1**	x_4	**0**	1	1	**1**

4	y_1	y_2	y_3	y_4	4	y_2	y_3	y_4	y_1
x_1	1	**1**	0	1	x_1	**1**	**0**	1	0
x_2	1	1	**1**	0	x_2	**1**	**1**	0	1
x_3	0	1	1	**1**	x_3	1	1	**1**	1
x_4	**1**	0	1	1	x_4	0	1	1	**1**

5	y_1	y_2	y_3	y_4	5	y_2	y_1	y_3	y_4
x_1	1	**1**	0	1	x_1	**1**	1	0	**1**
x_2	**1**	1	1	0	x_2	1	**1**	1	0
x_3	0	1	**1**	1	x_3	1	0	**1**	1
x_4	1	0	1	**1**	x_4	**0**	1	1	**1**

6	y_1	y_2	y_3	y_4	**6**	y_2	y_1	y_4	y_3
x_1	1	**1**	0	1	x_1	**1**	1	1	0
x_2	**1**	1	1	0	x_2	1	**1**	0	1
x_3	0	1	1	**1**	x_3	1	0	**1**	1
x_4	1	0	**1**	1	x_4	0	1	1	**1**

7	y_1	y_2	y_3	y_4	7	y_4	y_1	y_2	y_3
x_1	1	1	0	**1**	x_1	**1**	**1**	1	0
x_2	**1**	1	1	0	x_2	**0**	**1**	1	1
x_3	0	**1**	1	1	x_3	1	0	**1**	1
x_4	1	0	**1**	1	x_4	1	1	0	**1**

-

7	y_1	y_2	y_3	y_4	7	y_4	y_1	y_2	y_3
x_1	1	1	0	**1**	x_1	**1**	**1**	1	0
x_2	**1**	1	1	0	x_2	**0**	**1**	1	1
x_3	0	**1**	1	1	x_3	1	0	**1**	1
x_4	1	0	**1**	1	x_4	1	1	0	**1**

-

8	y_1	y_2	y_3	y_4	**8**	y_4	y_3	y_2	y_1
x_1	1	1	0	**1**	x_1	**1**	0	1	1
x_2	1	1	**1**	0	x_2	0	**1**	1	1
x_3	0	**1**	1	1	x_3	1	1	**1**	0
x_4	**1**	0	1	1	x_4	1	1	0	**1**

7	y_1	y_2	y_3	y_4	7	y_4	y_1	y_2	y_3
x_1	1	1	0	**1**	x_1	**1**	**1**	1	0
x_2	**1**	1	1	0	x_2	**0**	**1**	1	1
x_3	0	**1**	1	1	x_3	1	0	**1**	1
x_4	1	0	**1**	1	x_4	1	1	0	**1**

8	y_1	y_2	y_3	y_4	**8**	y_4	y_3	y_2	y_1
x_1	1	1	0	**1**	x_1	**1**	0	1	1
x_2	1	1	**1**	0	x_2	0	**1**	1	1
x_3	0	**1**	1	1	x_3	1	1	**1**	0
x_4	**1**	0	1	1	x_4	1	1	0	**1**

9	y_1	y_2	y_3	y_4	9	y_4	y_2	y_3	y_1
x_1	1	1	0	**1**	x_1	**1**	**1**	0	1
x_2	1	**1**	1	0	x_2	**0**	**1**	1	1
x_3	0	1	**1**	1	x_3	1	1	**1**	0
x_4	**1**	0	1	1	x_4	1	0	1	**1**

7	y_1	y_2	y_3	y_4	7	y_4	y_1	y_2	y_3
x_1	1	1	0	**1**	x_1	**1**	**1**	1	0
x_2	**1**	1	1	0	x_2	**0**	**1**	1	1
x_3	0	**1**	1	1	x_3	1	0	**1**	1
x_4	1	0	**1**	1	x_4	1	1	0	**1**

8	y_1	y_2	y_3	y_4	**8**	y_4	y_3	y_2	y_1
x_1	1	1	0	**1**	x_1	**1**	0	1	1
x_2	1	1	**1**	0	x_2	0	**1**	1	1
x_3	0	**1**	1	1	x_3	1	1	**1**	0
x_4	**1**	0	1	1	x_4	1	1	0	**1**

9	y_1	y_2	y_3	y_4	9	y_4	y_2	y_3	y_1
x_1	1	1	0	**1**	x_1	**1**	**1**	0	1
x_2	1	**1**	1	0	x_2	**0**	**1**	1	1
x_3	0	1	**1**	1	x_3	1	1	**1**	0
x_4	**1**	0	1	1	x_4	1	0	1	**1**

- $Perm\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} = 9 \qquad Sym\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} = 3$

- An **induced matching** in a graph G is a matching M where no two edges of M are joined by an edge.
- Every induced macthing in a bipartite graph is symmetric as well.
- Consequently, the size of a **maximum symmetric matching** is greater or equal to the size of a **maximum induced matching**.
- The problem of finding a **maximum induced matching** is NP-**hard**, even for bipartite graphs (K. Cameron, Discrete Applied Math, 1989; L. J. Stockmeyer and V. V. Vazirani, Inform. Proc. Letters, 1982).

Example

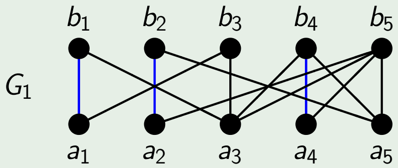

Figure: "Blue matchings" are **induced matchings**.

So Much for Today, but ...

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Problem

Estimate the number of symmetric perfect matchings of a balanced bipartite graph.

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Find the size of a maximum symmetric matching of a bipartite graph.

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Given a balanced bipartite graph without twins and a symmetric perfect matching, find another symmetric perfect macthing, if any.

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Problem

Given a balanced bipartite graph without twins and a symmetric perfect matching, find another symmetric perfect macthing, if any.

Conjecture

All square-roots of a König-Egerváry graph G are isomorphic.